Fractions

Fractions are numbers of the form

$$\frac{a}{b},$$

where a, b are numbers and $b \neq 0$. a is the *numerator* and b is the *denominator*.

Rules of Calculation

The following rules apply:

$$\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}, \quad \frac{a}{b}\frac{c}{d} = \frac{ac}{bd}, \quad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc},$$

$$a\frac{b}{c} = \frac{ab}{c}, \quad \frac{\frac{a}{b}}{c} = \frac{a}{bc}, \quad \frac{a}{\frac{b}{c}} = \frac{ac}{b}.$$

Reducing/Expanding Fractions

Common factors can be reduced:

$$\frac{a}{b} = \frac{ac}{bc}$$

Exponents

Exponents are numbers of the form x^a, where x is the *base* and a is the *exponent*.

Rules of Calculation

The following rules apply:

$$x^a x^b = x^{a+b}, \quad \frac{x^a}{x^b} = x^{a-b}, \quad (xy)^a = x^a y^a,$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}, \quad (x^a)^b = x^{ab}, \quad x^{-a} = \frac{1}{x^a}.$$

Roots

If $x \geq 0$ and $n \in \mathbf{Z}_+$ then there exists a number $\sqrt[n]{x} > 0$ such that

$$(\sqrt[n]{x})^n = x.$$

Note that $\sqrt[n]{x} = x^{\frac{1}{n}}$.

Rules of Calculation

The following rules apply:

$$\sqrt[n]{x} = x^{\frac{1}{n}}, \quad \sqrt[n]{x^m} = x^{\frac{m}{n}} = (\sqrt[n]{x})^m,$$

$$\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}, \quad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}.$$

Square Identities

The following rules apply:

$$(a+b)^2 = a^2 + b^2 + 2ab$$
$$(a-b)^2 = a^2 + b^2 - 2ab$$
$$(a+b)(a-b) = a^2 - b^2.$$

Equations

Equations can be simplified using the following rules:

1. You may add or subtract the same number on both sides of an equation.
2. You may multiply or divide both sides of an equation by the same number (except 0).

Quadratic Equations

Quadratic equations have the form

$$ax^2 + bx + c = 0,$$

The solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Factorization

If $ax^2 + bx + c = 0$ has roots r_1 and r_2, then the following holds:

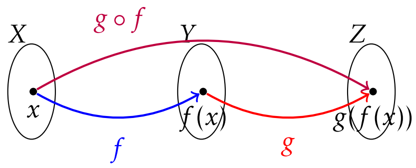
$$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

Functions

A function $f\colon X \to Y$ assigns to each $x \in X$ *exactly one* element $f(x) \in Y$.

Composite Functions

If $f\colon X \to Y$ and $g\colon Y \to Z$, then the composition $g \circ f\colon X \to Z$ is defined by $(g \circ f)(x) = g(f(x))$. f is the *inner function*, and g is the *outer function*.



Inverse Functions

Two functions $f\colon X \to Y$ and $g\colon Y \to X$ are *inverses* of each other if

$$f(g(y)) = y, \quad \text{and} \quad g(f(x)) = x$$

for all x in X and y in Y.

Polynomials

A first-degree polynomial is given by:

$$f(x) = ax + b.$$

A second-degree polynomial is given by:

$$f(x) = ax^2 + bx + c.$$

Logarithms and Exponential Functions

The *logarithm with base* a, $\log_a\colon]0, \infty[\to \mathbf{R}$, is the inverse of the exponential function $f_a(x) = a^x$ ($a > 0$, $a \neq 1$). It holds that

$$\log_a(a^x) = x \quad \text{and} \quad a^{\log_a(y)} = y$$

and we have

$$\ln x = \log_e x, \quad \log x = \log_{10} x$$

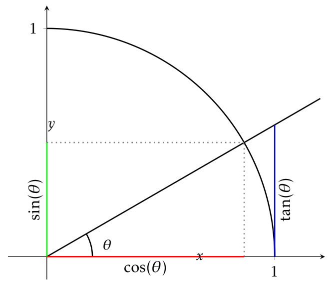
Rules of Calculation

The following rules apply:

$$\log_a(xy) = \log_a(x) + \log_a(y),$$
$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y),$$
$$\log_a(x^r) = r\log_a(x).$$

Trigonometric Functions

The trigonometric functions are defined based on the unit circle:



It holds that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ and:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	-

Differential Calculus

The derivative of f is written as $f' = \frac{d}{dx}f = \frac{df}{dx}$.

Rules of Differentiation

We have the following:

$f(x)$	$f'(x)$
c	0
x	1
x^n	nx^{n-1}
e^x	e^x
e^{cx}	ce^{cx}
a^x	$a^x \ln a$
$\ln x$	$\frac{1}{x}$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$1 + \tan^2(x)$

General Rules of Differentiation

The following hold:

$$(cf)'(x) = cf'(x)$$
$$(f \pm g)'(x) = f'(x) \pm g'(x)$$
$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$
$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$
$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

The last rule is called the *chain rule*.

Indefinite Integrals

A function f has an *antiderivative* F if

$$F'(x) = f(x).$$

The indefinite integral of f is

$$\int f(x)\,dx = F(x) + k,$$

where $F'(x) = f(x)$ and $k \in \mathbf{R}$.

General Rules

$$\int cf(x)\,dx = c\int f(x)\,dx$$
$$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$
$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx$$
$$\int f(g(x))g'(x)\,dx = F(g(x)) + k$$

The third rule is called *integration by parts*, and the last is called *integration by substitution*.

Common Rules

We have the following:

$f(x)$	$\int f(x)\,dx$
c	$cx + k$
x	$\frac{1}{2}x^2 + k$
x^n	$\frac{1}{n+1}x^{n+1} + k$
e^x	$e^x + k$
e^{cx}	$\frac{1}{c}e^{cx} + k$
$\frac{1}{x}$	$\ln(\lvert x\rvert) + k$
$\ln x$	$x\ln(x) - x + k$
$\cos x$	$\sin x + k$
$\sin x$	$-\cos x + k$
$\tan x$	$-\ln(\lvert\cos(x)\rvert) + k$

Integration by Substitution

Given an integral of the form $\int f(g(x))g'(x)\,dx$, apply the following method:

1. Let $u = g(x)$.
2. Compute $\frac{du}{dx}$ and isolate dx.
3. Substitute $g(x)$ and dx.
4. Evaluate the integral with respect to u.
5. Substitute back.

Definite Integrals

The definite integral of f over the interval $[a,b]$ is

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a),$$

where F is an antiderivative of f.

General Rules

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$
$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$
$$\int_a^b f(x)g(x)\,dx = [f(x)G(x)]_a^b - \int_a^b f'(x)G(x)\,dx$$
$$\int_a^b f(g(x))g'(x)\,dx = [F(x)]_{g(a)}^{g(b)}$$

Integration by Substitution

Given an integral of the form $\int_a^b f(g(x))g'(x)\,dx$, apply the following method:

1. Let $u = g(x)$.
2. Compute $\frac{du}{dx}$ and isolate dx.
3. Substitute $g(x)$, dx and the bounds.
4. Evaluate the integral with respect to u.

Differential Equations

Solution Formulas

Differential Equation	General Solution
$f'(x) = k$	$f(x) = kx + c$
$f'(x) = h(x)$	$f(x) = \int h(x)\,dx$
$f'(x) = kf(x)$	$f(x) = ce^{kx}$
$f'(x) + af(x) = b$	$f(x) = \frac{b}{a} + ce^{-ax}$

The Armor Formula

The differential equation

$$f'(x) + a(x)f(x) = b(x)$$

has the general solution

$$f(x) = e^{-A(x)}\int b(x)e^{A(x)}\,dx + ce^{-A(x)},$$

where $A'(x) = a(x)$.

Vectors in the Plane

A vector $\vec{u}$ in the plane is written as $\vec{u} = [x,y]$ where $x, y \in \mathbf{R}$.

Rules of Calculation

For $\vec{u} = [x_1, y_1]$, $\vec{v} = [x_2, y_2]$, and $c \in \mathbf{R}$, we have:

$$\vec{u} \pm \vec{v} = \begin{bmatrix} x_1 \pm x_2 \\ y_1 \pm y_2 \end{bmatrix}, \qquad \vec{u}\cdot\vec{v} = x_1x_2 + y_1y_2,$$
$$c\vec{u} = \begin{bmatrix} cx_1 \\ cy_1 \end{bmatrix}, \quad \det(\vec{u},\vec{v}) = x_1y_2 - x_2y_1$$

The length of $\vec{u}$ is $\lVert\vec{u}\rVert = \sqrt{x_1^2 + y_1^2}$.

The Angle Between Two Vectors

For the angle θ between $\vec{u}$ and $\vec{v}$, we have:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\lVert\vec{u}\rVert\,\lVert\vec{v}\rVert}, \quad \sin\theta = \frac{\det(\vec{u},\vec{v})}{\lVert\vec{u}\rVert\,\lVert\vec{v}\rVert}$$

Furthermore:

1. $\vec{u}$ and $\vec{v}$ are orthogonal $\Leftrightarrow \vec{u}\cdot\vec{v} = 0$.
2. $\vec{u}$ and $\vec{v}$ are parallel $\Leftrightarrow \det(\vec{u},\vec{v}) = 0$.

Vectors in Space

A vector $\vec{u}$ in space is written as $\vec{u} = [x,y,z]$ where $x, y, z \in \mathbf{R}$.

Rules of Calculation

For $\vec{u} = [x_1, y_1, z_1]$, $\vec{v} = [x_2, y_2, z_2]$, and $c \in \mathbf{R}$, we have:

$$\vec{u} \pm \vec{v} = \begin{bmatrix} x_1 \pm x_2 \\ y_1 \pm y_2 \\ z_1 \pm z_2 \end{bmatrix}, \qquad c\vec{u} = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix},$$
$$\vec{u}\cdot\vec{v} = x_1x_2 + y_1y_2 + z_1z_2.$$

The length of $\vec{u}$ is $\lVert\vec{u}\rVert = \sqrt{x_1^2 + y_1^2 + z_1^2}$. The cross product is given by:

$$\vec{u}\times\vec{v} = \begin{bmatrix} y_1z_2 - z_1y_2 \\ z_1x_2 - x_1z_2 \\ x_1y_2 - y_1x_2 \end{bmatrix}$$

The Angle Between Two Vectors

For the angle θ between $\vec{u}$ and $\vec{v}$, we have:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\lVert\vec{u}\rVert\,\lVert\vec{v}\rVert}, \quad \sin\theta = \frac{\lVert\vec{u}\times\vec{v}\rVert}{\lVert\vec{u}\rVert\,\lVert\vec{v}\rVert}$$

Furthermore:

1. $\vec{u}$ and $\vec{v}$ are orthogonal $\Leftrightarrow \vec{u}\cdot\vec{v} = 0$.
2. $\vec{u}$ and $\vec{v}$ are parallel $\Leftrightarrow \vec{u}\times\vec{v} = 0$.

Lines and Planes

The plane/line through the point with position vector $\vec{x_0}$ and normal vector $\vec{n}$ is described by all vectors $\vec{x}$ that solve the equation:

$$\vec{n}\cdot(\vec{x} - \vec{x_0}) = 0$$

A line in space or in the plane through the point with position vector $\vec{x_0}$ and direction vector $\vec{r}$ has the parametric representation:

$$\vec{x_0} + t\vec{r}, \quad t \in \mathbf{R}.$$